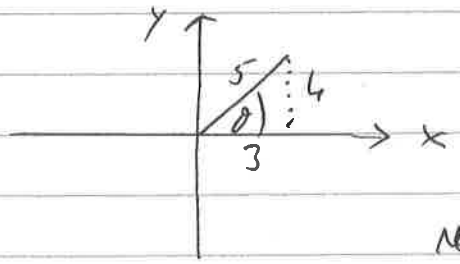
Mathematics: The Core Course for A-level; L. Bostock and S. Chandler

Chapter 7 : Trigonometric Identities

Exercise 7f

(1) (a) $\tan\theta = \frac{4}{3}$ & θ is acute : $0 \le \theta \le 90°$



$0 \le \theta \le 90°$

$\Rightarrow \quad 0 \le 2\theta \le 180°$: quad I & II

Now, $\sin 2\theta = 2\sin\theta\cos\theta$

Since θ is acute $\cos\theta$ is positive

$\therefore \quad \sin 2\theta = 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}$

(b) $0 \le \theta \le 90° \Rightarrow 0 \le \frac{\theta}{2} \le 45°$ ie Still in quadrant I

Now, $\tan\frac{\theta}{2} = \frac{\sin\theta/2}{\cos\theta/2}$

But $\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow \cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$

$\therefore \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$

and $\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$\therefore \cos\frac{\theta}{2} = \sqrt{\frac{\cos\theta + 1}{2}}$

Hence $\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

$= \sqrt{\frac{1-3/5}{1+3/5}} = \sqrt{\frac{2/5}{8/5}} = \frac{1}{2}$

(c) $\cot 2\theta = \frac{1}{\tan 2\theta} = \frac{\cos 2\theta}{\sin 2\theta}$

$= \frac{2\cos^2 - 1}{2\sin\theta\cos\theta}$

θ is acute so $\cos\theta$ & $\sin\theta$ are Positive.

So $\cot 2\theta = \frac{2(3/5)^2 - 1}{2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)} = -\frac{7}{24}$

(Note: if θ is acute, $0^\circ \le \theta \le 90^\circ$. Then $0^\circ \le 2\theta \le 180^\circ$.

But $\cot 2\theta$ is Negative, hence θ lies in $90^\circ \le 2\theta \le 180^\circ$)

(2)

(a) $$\frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$$

$$= \frac{1+t^2-(1-t^2)}{1+t^2+1-t^2}$$

$$= \frac{2t^2}{2} = t^2$$

(b) $$\frac{\sin\theta}{1-\cos\theta} = \frac{2t}{1+t^2} \div \left(1-\frac{1-t^2}{1+t^2}\right)$$

$$= \frac{2t}{1+t^2} \div \left(\frac{1+t^2-(1-t^2)}{1+t^2}\right)$$

$$= \frac{2t}{1+t^2} \cdot \frac{1+t^2}{2t^2}$$

$$= \frac{1}{t}$$

(c) $$\cot\theta \cot\frac{\theta}{2} = \frac{\cos\theta}{\sin\theta} \cdot \frac{\cos\theta/2}{\sin\theta/2}$$

or $$= \frac{1}{\tan\theta} \cdot \frac{1}{\tan\frac{\theta}{2}}$$

$$= \frac{1-t^2}{2t} \cdot \frac{1}{t} = \frac{1-t^2}{2t^2}$$

(d) $$\frac{\cos^2 \theta/2}{3\sin\theta + 4\cos\theta - 1}$$

By The trig identity $\cos 2A = 2\cos^2 A - 1$
we have $\cos A = 2\cos^2 \frac{A}{2} - 1$

$$\therefore \quad \cos^2 \frac{A}{2} = \frac{\cos A + 1}{2} = \frac{\frac{1-t^2}{1+t^2} + 1}{2}$$

hence we have $$\frac{\frac{1}{2}\left(\frac{1-t^2}{1+t^2} + 1\right)}{3 . \frac{2t}{1+t^2} + 4 . \frac{1-t^2}{1+t^2} - 1}$$

$$= \frac{1}{2} . \frac{1-t^2 + 1 + t^2}{6t + 4(1-t^2) - (1+t^2)}$$

$$= \frac{1}{-5t^2 + 6t + 3}$$

(e) $$\frac{1 - 2\sin\theta}{2\cos\theta + 1} = \frac{1 - 2\,(2t/(1+t^2))}{2(1-t^2)/(1+t^2) + 1}$$

$$= \frac{1 + t^2 - 4t}{3 - t^2}$$

$$= \frac{t^2 - 4t + 1}{3 - t^2}$$

(3) $\operatorname{cosec} A + \cot A \equiv \cot \frac{A}{2}$

$\therefore \quad \frac{1}{\sin A} + \frac{\cos A}{\sin A} \equiv \frac{1}{\tan \frac{A}{2}}$

LHS: $\quad \frac{1+t^2}{2t} + \frac{\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}} = \frac{1+t^2}{2t} + \frac{1-t^2}{2t}$

$= \frac{2}{2t} = \frac{1}{t} \equiv \frac{1}{\tan \frac{A}{2}}$ ✓

(4) $\sec \theta - \tan \theta = x$

$\therefore \quad \frac{1}{\cos \theta} - \tan \theta = x$

So LHS: $\frac{1+t^2}{1-t^2} - \frac{2t}{1-t^2} = \frac{t^2 - 2t + 1}{1-t^2}$

$= \frac{(t-1)^2}{1-t^2} = x$

So

$\tan \frac{\theta}{2} = \frac{1-x}{1+x} = \frac{1 - \frac{(t-1)^2}{1-t^2}}{1 + \frac{(t-1)^2}{1-t^2}}$

$= \frac{1-t^2-(t-1)^2}{1-t^2+(t-1)^2}$

$= \frac{-2t^2+2t}{2-2t} = \frac{t-t^2}{1-t}$

$= \frac{t(1-t)}{1-t} = t$ ✓

(5) (a) $3\cos\theta + 2\sin\theta = 3$

$\therefore \quad 3\dfrac{(1-t^2)}{1+t^2} + 2 \cdot \dfrac{2t}{1+t^2} = 3$

So $\quad 3(1-t^2) + 4t = 3(1+t^2)$

$\therefore \quad 6t^2 - 4t = 0$

$\therefore \quad t(3t-2) = 0$

$\Rightarrow \quad t = \tan\dfrac{\theta}{2} = 0 \quad$ and $\quad t = \tan\dfrac{\theta}{2} = \dfrac{2}{3}$

For $\tan\dfrac{\theta}{2} = 0$: $\quad \dfrac{\theta}{2} = 0^\circ \pm 180^\circ n$

$\theta = \pm 360^\circ n \quad , n = 0, 1, 2, \ldots$

So in $-180^\circ \leq \theta \leq +180^\circ$, $\theta = 0^\circ$

For $\tan\dfrac{\theta}{2} = \dfrac{2}{3}$: $\quad \dfrac{\theta}{2} = 33.69^\circ \pm 180^\circ n$

So $\theta = 67.38^\circ \pm 360^\circ n \quad , n = 0, 1, 2, \ldots$

So in $-180^\circ \leq \theta \leq 180^\circ$, $\theta = 67.38^\circ$

(b) $5\cos\theta - \sin\theta + 4 = 0$

So $5 \cdot \frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} + 4 = 0$

$\therefore \quad 5(1-t^2) - 2t + 4(1+t^2) = 0$

$t^2 + 2t - 9 = 0$

So $t = \frac{-2 \pm \sqrt{4+36}}{2} = -1 \pm \sqrt{10}$

$\therefore$

$\tan\frac{\theta}{2} = -1 + \sqrt{10}$ or $\tan\frac{\theta}{2} = -1 - \sqrt{10}$

$\therefore \quad \frac{\theta}{2} = 65.18^\circ \pm 180^\circ n \;; \quad \frac{\theta}{2} = -76.49^\circ \pm 180^\circ n$

$\theta = 130.36^\circ \pm 360^\circ n \;; \quad \theta = -152.98^\circ \pm 360^\circ n$

So in $-180^\circ \le \theta \le 180^\circ$, $\theta = 130.36^\circ, -152.98^\circ$

(c) $\cos\theta + 7\sin\theta = 5$

So $\frac{1-t^2}{1+t^2} + 7 \cdot \frac{2t}{1+t^2} = 5$

$\therefore \quad 1 - t^2 + 14t = 5(1+t^2)$

$\Rightarrow \quad 3t^2 - 7t + 2 = 0$

$(3t-1)(t-2) = 0 \quad \Rightarrow \quad t = \frac{1}{3}$ or 2

$\therefore \quad \tan\frac{\theta}{2} = \frac{1}{3} \quad$ or $\quad \tan\frac{\theta}{2} = 2$

So $\frac{\theta}{2} = 18.43° \pm 180°n$, $\frac{\theta}{2} = 63.43° \pm 180°n$

$\therefore \quad \theta = 36.87° \pm 360°n$, $\theta = 126.87° \pm 360°n$

So in $-180° \leq \theta \leq 180°$, $\theta = 36.87°$, $126.89°$

(d) $2\cos\theta - \sin\theta = 1$

$\therefore \quad 2 \cdot \frac{(1-t^2)}{1+t^2} - \frac{2t}{1+t^2} = 1$

$\therefore \quad 2(1-t^2) - 2t = 1+t^2$

$\Rightarrow \quad 3t^2 + 2t - 1 = 0$

$(3t-1)(t+1) = 0 \Rightarrow t = \frac{1}{3}$ or -1

$\therefore \quad \tan\frac{\theta}{2} = \frac{1}{3} \quad$ or $\quad \tan\frac{\theta}{2} = -1$

So $\frac{\theta}{2} = 18.43° \pm 180°n \quad$ OR $\quad \frac{\theta}{2} = -45° \pm 180°n$

$\theta = 36.87° \pm 360°n \qquad \theta = -90° \pm 360°n$

So in $-180° \leq \theta \leq 180°$, $\theta = 36.87°$, $-90°$

(6) (a) $\sqrt{3}\cos\theta - \sin\theta = r\cos(\theta+\alpha)$

$= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$

Compare LHS & RHS: $\sqrt{3} = r\cos\alpha$

$-1 = -r\sin\alpha$

i) Square & add: $(\sqrt{3})^2 + (+1)^2 = r^2\cos^2\alpha + (+r)^2\sin^2\alpha$

$3 + 1 = r^2(\cos^2\alpha + \sin^2\alpha)$

$r = \pm 2$

But r is always taken to be the positive value (since it is the length of the hypotenuse of the triangle), so $r = 2$

ii) Divide: $\dfrac{-1}{\sqrt{3}} = \dfrac{r\sin\alpha}{r\cos\alpha} = \tan\alpha$

$\therefore \alpha = \tan^{-1}\dfrac{-1}{\sqrt{3}} = -30^\circ$

So

$\sqrt{3}\cos\theta - \sin\theta = 2\cos(\theta - 30^\circ)$

(b) $\cos\theta + 3\sin\theta = r\cos(\theta-\alpha)$

$= r\cos\theta\cos\alpha + r\sin\theta\sin\alpha$

Compare LHS & RHS: $1 = r\cos\alpha$

$3 = r\sin\alpha$

i) Square & add: $1^2 + 3^2 = r^2\cos^2\alpha + r^2\sin^2\alpha$

$\Rightarrow 10 = r^2, \therefore r = \pm\sqrt{10}$

But r is always taken as positive (since it is the length of the hypotenuse of the triangle), $\therefore$ $r = \sqrt{10}$

ii) divide : $\frac{3}{1} = \frac{r \sin\alpha}{r\cos\alpha} = \tan\alpha$

$\therefore \quad \alpha = \tan^{-1} 3 = 71.57^\circ$

$\therefore \quad \cos\theta + 3\sin\theta = \sqrt{10}\cos(\theta - 71.57^\circ)$

(c) $4\sin\theta - 3\cos\theta = r\sin(\theta - \alpha)$

$= r\sin\theta\cos\alpha - r\sin\alpha\cos\theta$

Compare LHS & RHS : $4 = r\cos\alpha$
$-3 = -r\sin\alpha$

i) Square & add : $4^2 + 3^2 = r^2\cos^2\alpha + r^2\sin^2\alpha$

$16 + 9 = r^2(\cos^2\alpha + \sin^2\alpha)$
$25 = r^2$

So $r = \pm 5$

But r is always taken as positive since it is the length of the hyp of a triangle. So $r = 5$

ii) Divide : $\frac{3}{4} = \frac{r \sin\alpha}{r\cos\alpha} = \tan\alpha$

So $\alpha = \tan^{-1}\frac{3}{4} = 36.87^\circ$

So $4\sin\theta - 3\cos\theta = 5\sin(\theta - 36.87)$

(d) $\cos 2\theta - \sin 2\theta = r\cos(2\theta + \alpha)$

$= r\cos 2\theta\cos\alpha - r\sin 2\theta\sin\alpha$

Compare LHS & RHS : $1 = r\cos\alpha$
$1 = r\sin\alpha$

i) Square & add : $1^2 + 1^2 = r^2\cos^2\alpha + r^2\sin^2\alpha$

So $1 = r^2 \Rightarrow r = \pm 1$

But r is always positive since it is the length of the hypotenuse of a triangle. So $r = 1$

ii) Divide : $\frac{1}{1} = \frac{r\sin\alpha}{r\cos\alpha} = \tan\alpha$

$\therefore \alpha = \tan^{-1} 1 = 45^\circ$

So $\cos 2\theta - \sin 2\theta = \cos(2\theta + 45^\circ)$

(e) $2\cos 3\theta + 5\sin 3\theta = r\sin(3\theta + \alpha)$

$= r\sin 3\theta\cos\alpha + r\sin\alpha\cos 3\theta$

Compare LHS & RHS: $2 = r\sin\alpha$

$5 = r\cos\alpha$

i) Square & add: $2^2 + 5^2 = r^2\sin^2\alpha + r^2\cos^2\alpha$

$29 = r^2$

So $r = \pm\sqrt{29}$

But r is always positive because it is the length of the hypotenuse of a triangle. So $r = \sqrt{29}$

ii) Divide: $\frac{2}{5} = \frac{r\sin\alpha}{r\cos\alpha} = \tan\alpha$

$\therefore \alpha = \tan^{-1}\frac{2}{5} = 21.8^\circ$

$\therefore 2\cos 3\theta + 5\sin 3\theta = \sqrt{29}\sin(3\theta + 21.8^\circ)$

(7) (a) let $\cos\theta - \sqrt{3}\sin\theta = r\cos(\theta+\alpha)$

$\therefore \quad \cos\theta - \sqrt{3}\sin\theta = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$

Compare LHS & RHS: $\quad 1 = r\cos\alpha$

$\sqrt{3} = r\sin\alpha$

i) Square & add: $\quad 1^2 + (\sqrt{3})^2 = r^2\cos^2\alpha + r^2\sin^2\alpha$

So $\quad 4 = r^2 \Rightarrow r = \pm 2$

$\Rightarrow r = 2$

ii) Divide:

$$\frac{r\sin\alpha}{r\cos\alpha} = \frac{\sqrt{3}}{1} = \tan\alpha$$

So $\alpha = \tan^{-1}\sqrt{3} = 60°$

$\therefore$

$$\cos\theta - \sqrt{3}\sin\theta = 2\cos(\theta + 60°)$$

Now, cos is max at 1 & min at -1. So

$\cos\theta - \sqrt{3}\sin\theta$ is max at 2 & min at -2.

For value of θ at max & min, no need for calculus: in $0° \to 360°$

$\cos\theta$ is max at $\theta = 0°$ and $360°$

$\cos(\theta + 60)$ implies a left shift by $60°$ so max occurs at $\theta = -60°$ & $300°$.

we only want θ in $[0°, 360°]$, so max is at $\theta = 300°$

$\cos\theta$ is min at $\theta = 180°$

$\cos(\theta + 60)$ implies a left shift of $60°$ so min occurs at $\theta = 120$.

(b) $7\cos\theta - 24\sin\theta + 3$.

let $7\cos\theta - 24\sin\theta = r\cos(\theta + \alpha)$

$= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$

So $7 = r\cos\alpha$
and $24 = r\sin\alpha$

i) Square & add : $7^2 + 24^2 = r^2\cos^2\alpha + r^2\sin^2\alpha$

So $r = \pm 25 \Rightarrow r = 25$

ii) divide : $\dfrac{24}{7} = \dfrac{r\sin\alpha}{r\cos\alpha} = \tan\alpha$

So $\alpha = 73.74°$

So $7\cos\theta - 24\sin\theta + 3 = 25\cos(\theta + 73.74°) + 3$

Now, $\cos$ is max at $+1$ & min at -1

So $25\cos(\theta + 73.74) + 3$ is max at 28 & min at -22

For values of θ at max / min in $0° \to 360°$

$\cos\theta$ is max at $0°$ and $360°$

$\cos(\theta + 73.74)$ is a left-shift by $73.74°$ so max occurs at $\theta = -73.74°$ and $286.26°$

But we want θ only in $[0°, 360°]$ so $\theta = 286.26°$

$\cos\theta$ is min at $\theta = 180°$, but this is shifted left by $73.74°$
So min occurs at $\theta = 106.26°$

(c) $\dfrac{1}{\cos 2\theta + \sin 2\theta}$

Let $\cos 2\theta + \sin 2\theta = r\sin(2\theta + \alpha)$

$$= r\sin 2\theta\cos\alpha + r\sin\alpha\cos 2\theta.$$

Compare LHS & RHS : $1 = r\cos\alpha$
$1 = r\sin\alpha$

i) Square & add: $1^2 + 1^2 = r^2\cos^2\alpha + r^2\sin^2\alpha$

$$\therefore r^2 = \pm\sqrt{2} \Rightarrow r = \sqrt{2}$$

ii) divide $\dfrac{1}{1} = \dfrac{r\sin\alpha}{r\cos\alpha} = \tan\alpha$

So $\alpha = \tan^{-1} 1 = 45°$

$$\therefore \quad \frac{1}{\cos 2\theta + \sin 2\theta} = \frac{1}{\sqrt{2}\,\sin(2\theta + 45°)}$$

Now, Sin is max at +1 & min at −1

So $\frac{1}{\sin}$ is max at −1 & min at +1

$\therefore \frac{1}{\sqrt{2}\,\sin(2\theta+45)}$ is max $-\frac{1}{\sqrt{2}}$ & min at $\frac{1}{\sqrt{2}}$

Now, Sin θ is max at $\theta = 90°$ for θ in $[0; 360°]$

So $2\theta + 45 = 90 \Rightarrow \theta = 22.5°$
<u>and</u> $2\theta + 45 = 450 \Rightarrow \theta = 202.5°$

Don't forget this last answer: 450° may lie outside 360° but the θ which gives 450° lies inside 360°.

Also Sinθ is min at $\theta = 270°$ for θ in $[0; 360°]$

So $2\theta + 45 = 270 \Rightarrow \theta = 112.5°$
<u>and</u> $2\theta + 45 = 630 \Rightarrow \theta = 292.5°$

Again, don't forget this last answer. The key is to test a few angles outside the Range to see if θ lies inside the Range.

(d) $\dfrac{\sqrt{2}}{\cos\theta - \sqrt{2}\sin\theta}$

Let $\cos\theta - \sqrt{2}\sin\theta = r\sin(\theta - \alpha)$

$= r\sin\theta\cos\alpha - r\sin\alpha\cos\theta$

Compare LHS & RHS : $1 = -r\sin\alpha$
$-\sqrt{2} = r\cos\alpha$

i) Square & add : $1^2 + (-\sqrt{2})^2 = (-r\sin\alpha)^2 + (r\cos\alpha)^2$

$3 = r^2(\cos^2\alpha + \sin^2\alpha)$

So $r = \pm\sqrt{3} \Rightarrow r = \sqrt{3}$

(Since r is always Positive)

ii) Divide : $\dfrac{-1}{-\sqrt{2}} = \dfrac{r\sin\alpha}{r\cos\alpha} = \tan\alpha$

So $\alpha = \tan^{-1}\dfrac{1}{\sqrt{2}} = 35.26^\circ$

So $\dfrac{\sqrt{2}}{\cos\theta - \sqrt{2}\sin\theta} = \dfrac{\sqrt{2}}{\sqrt{3}\sin(\theta - 35.26)}$

Now Sin is max at +1 and min at −1

So $\dfrac{1}{\sin}$ is max at −1 and min at +1

$\therefore \quad \dfrac{\sqrt{2}}{\sqrt{3}\,\sin(\theta - 35.26)}$ is max at $-\dfrac{\sqrt{2}}{\sqrt{3}}$

and min at $+\dfrac{\sqrt{2}}{\sqrt{3}}$

Also, $\sin$ is max at $\theta = 90°$ (in $[0, 360°]$)

So $\dfrac{\sqrt{2}}{\sqrt{3}\,\sin(\theta - 35.26)}$ is max when $\theta - 35.26 = 90°$

i.e $\theta = 125.26°$

$\sin$ is min at $\theta = 270$ (in $[0, 360]$)

So $\dfrac{\sqrt{2}}{\sqrt{3}\,\sin(\theta - 35.26)}$ is min when $\theta - 35.26 = 270°$

i.e $\theta = 305.26°$

(e) $(3\cos\theta + 4\sin\theta)^2 = 9\cos^2\theta + 24\cos\theta\sin\theta + 16\sin^2\theta$

$= 9\cos^2\theta + 16(1 - \cos^2\theta) + 24\cos\theta\sin\theta$

$= -7\cos^2\theta + 16 + 12\sin 2\theta$

By $\cos 2\theta = 2\cos^2\theta - 1$ we have $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\therefore \quad (3\cos\theta + 4\sin\theta)^2 = -\frac{7}{2}\cos 2\theta - \frac{7}{2} + 16 + 12\sin 2\theta$

$= -\frac{7}{2}\cos 2\theta + 12\sin 2\theta + \frac{25}{2}$

let $\quad -\frac{7}{2}\cos 2\theta + 12\sin 2\theta = r\sin(2\theta + \alpha)$

$= r\sin 2\theta\cos\alpha + r\sin\alpha\cos 2\theta$

Compare LHS & RHS: $\quad -\frac{7}{2} = r\sin\alpha$

$12 = r\cos\alpha$

i) Square & add: $\quad \left(-\frac{7}{2}\right)^2 + (12)^2 = r^2(\cos^2\alpha + \sin^2\alpha)$

$\therefore \quad \frac{49}{4} + 144 = r^2$

$\Rightarrow r = \pm 12.5 \quad \Rightarrow \quad r = 12.5$

since r is always positive

ii) Divide: $\quad \frac{-7/2}{12} = \frac{r\sin\alpha}{r\cos\alpha} = \tan\alpha$

$-\frac{7}{24} = \tan\alpha, \quad \therefore \quad \alpha = \tan^{-1}\left(\frac{-7}{24}\right)$

$= -16.26°$

So $\quad (3\cos\theta + 4\sin\theta)^2 = 12.5\sin(2\theta - 16.26°) + 12.5$

Now, $\sin$ is max at $+1$ & min at -1

So $12.5 \sin(2\theta - 16.26^\circ) + 12.5$ is max at $12.5 + 12.5 = 25$
& min at $12.5 - 12.5 = 0$

Also $\sin$ is max when $\theta = 90^\circ$ (in $[0, 360]$)

So $2\theta - 16.26 = 90^\circ \Rightarrow \theta = 53.13^\circ$
But also $2\theta - 16.26 = 450^\circ \Rightarrow \theta = 233.13^\circ$

Don't forget this last answer: 450° lies outside 360°, but the θ which gives 450° lies inside 360°.

And ... $\sin$ is min when $\theta = 270^\circ$ (in $[0, 360]$)

So $2\theta - 16.26 = 270 \Rightarrow \theta = 143.13^\circ$
But also $2\theta - 16.26 = 630 \Rightarrow \theta = 323.13^\circ$

8 a $\cos x + \sin x = \sqrt{2}$

let $r \sin(x + \alpha) = \cos x + \sin x$

$\therefore$

$r \sin x \cos\alpha + r \sin\alpha \cos x = \cos x + \sin x$

compare LHS & RHS: $r \sin\alpha = 1$
$r \cos\alpha = 1$

i) Square & add: $r^2(\cos^2\alpha + \sin^2\alpha) = 1^2 + 1^2$

$\therefore r^2 = 1 \Rightarrow r = \pm\sqrt{2} \Rightarrow r = \sqrt{2}$, since r is always positive.

ii) Divide $\frac{6}{7} = \frac{r\sin\alpha}{r\cos\alpha} = \tan\alpha$

So $\alpha = \tan^{-1}\frac{6}{7} = 40.6^\circ$

So $7\cos x + 6\sin x = \sqrt{85}\cos(x - 40.6^\circ) = 2$

$\therefore \cos(x - 40.6^\circ) = \frac{2}{\sqrt{85}}$

$x - 40.6^\circ = 77.47^\circ \pm 360^\circ n$

and $x - 40.6^\circ = -77.47^\circ \pm 360^\circ n$

$\therefore x = 118.07^\circ \pm 360^\circ n$

and $x = 36.87^\circ \pm 360^\circ n$

(c) $\cos x - 3\sin x = 1$

Let $\cos x - 3\sin x = r\sin(x - \alpha)$

$= r\sin x\cos\alpha - r\sin\alpha\cos x$

Compare LHS & RHS: $1 = -r\sin\alpha$

$-3 = r\cos\alpha$

i) Square & add: $1 + 9 = r^2\sin^2\alpha + r^2\cos^2\alpha$

So $10 = r^2 \Rightarrow r = \pm\sqrt{10} = \sqrt{10}$

since r is always positive.

ii) Divide: $\dfrac{r\sin\alpha}{r\cos\alpha} = \tan\alpha = 1$

So $\alpha = \tan^{-1} 1 = 45^\circ$

So $\cos x + \sin x = \sin(\theta + 45)$

$\therefore \sqrt{2}\sin(\theta + 45^\circ) = \sqrt{2}$

$\Rightarrow \theta + 45^\circ = 90^\circ \pm 360^\circ n$

$\therefore \theta = 45^\circ \pm 360^\circ n = \dfrac{\pi}{4} \pm 2n\pi$

(b) $7\cos x + 6\sin x = 2$

Let $7\cos x + 6\sin x = r\cos(x - \alpha)$

$= r\cos x\cos\alpha + r\sin x\sin\alpha$

Compare LHS & RHS: $7 = r\cos\alpha$
$6 = +r\sin\alpha$

i) Square & add: $49 + 36 = r^2\cos^2\alpha + r^2\sin^2\alpha$

$85 = r^2 \Rightarrow r = \pm\sqrt{85}$

$= \sqrt{85}$

Since r is always positive.

ii) Divide : $\frac{1}{3} = \frac{r \sin\alpha}{r\cos\alpha} = \tan\alpha$

So $\alpha = \tan^{-1}\frac{1}{3} = 18.43^\circ$

hence $\cos x - 3\sin x = \sqrt{10}\,\sin(x - 18.43^\circ) = 1$

$\therefore \sin(x - 18.43^\circ) = \frac{1}{\sqrt{10}}$

$x - 18.43^\circ = 18.43^\circ \pm 360^\circ n$

and $x - 18.43^\circ = 161.57^\circ \pm 360^\circ n$

Therefore $x = 36.87 \pm 360^\circ n$

and $x = 180^\circ \pm 360^\circ n$ (Book Answer is wrong)

(d) $2\cos x - \sin x = 2$

Let $2\cos x - \sin x = r\cos(x + \alpha)$

$= r\cos x\cos\alpha - r\sin x\sin\alpha$

Compare LHS & RHS: $2 = r\cos\alpha$

$-1 = -r\sin\alpha$

i) Square & add : $4 + 1 = r^2\cos^2\alpha + r^2\sin^2 a$

$5 = r^2 \Rightarrow r = \pm\sqrt{5} = \sqrt{5}$

Since r is always positive.

ii) Divide : $\frac{1}{2} = \frac{r \sin\alpha}{r \cos\alpha} = \tan\alpha$

So $\alpha = \tan^{-1}\frac{1}{2} = 26.57^\circ$

So $2\cos x - \sin x = \sqrt{5}\cos(x + 26.57) = 2$

$\therefore \quad \cos(x + 26.57) = \frac{2}{\sqrt{5}}$

$x + 26.57 = 26.57 \pm 360^\circ n$

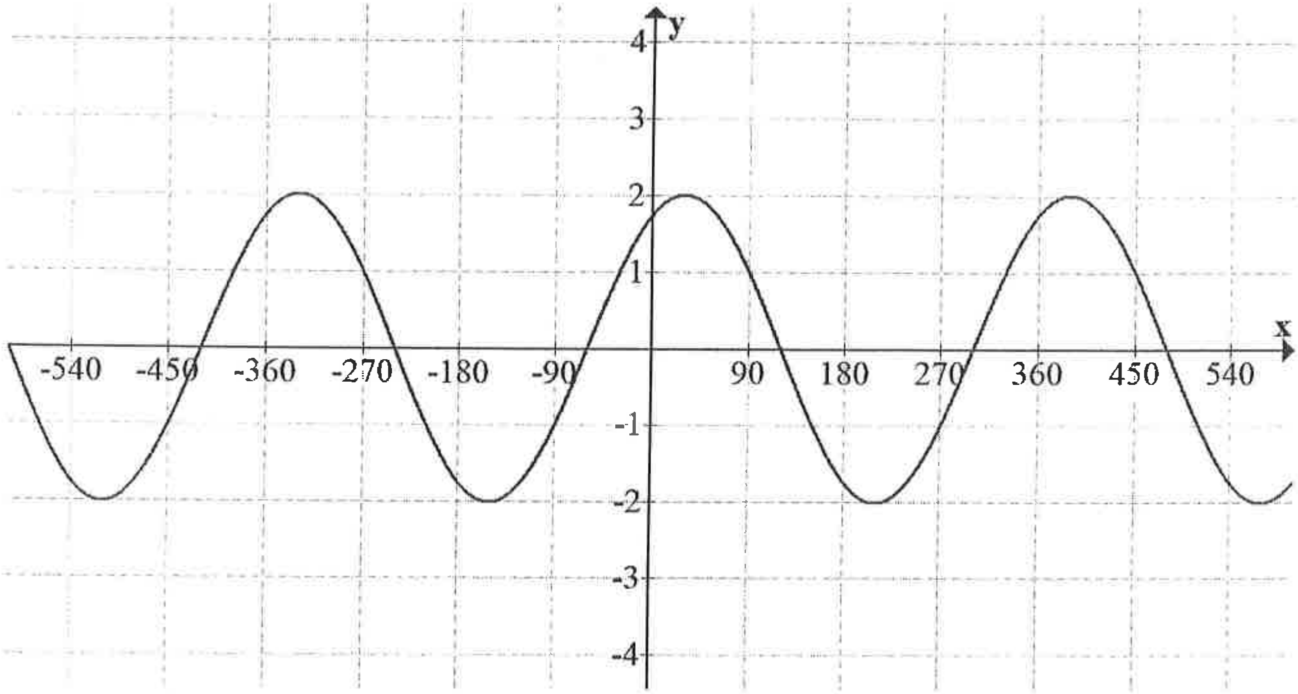
and $x + 26.57 = -26.57 \pm 360^\circ n$

$\therefore \quad x = \pm 360^\circ n$

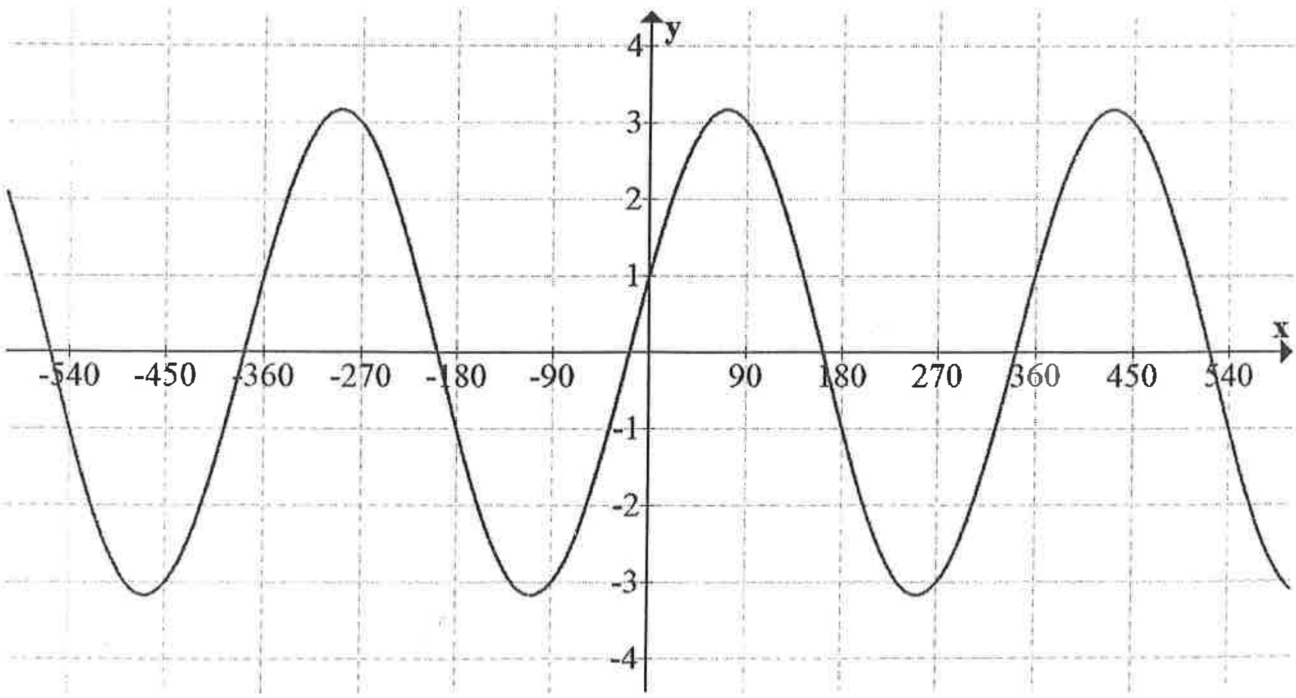
and $x = -53.13^\circ \pm 360^\circ n$

9

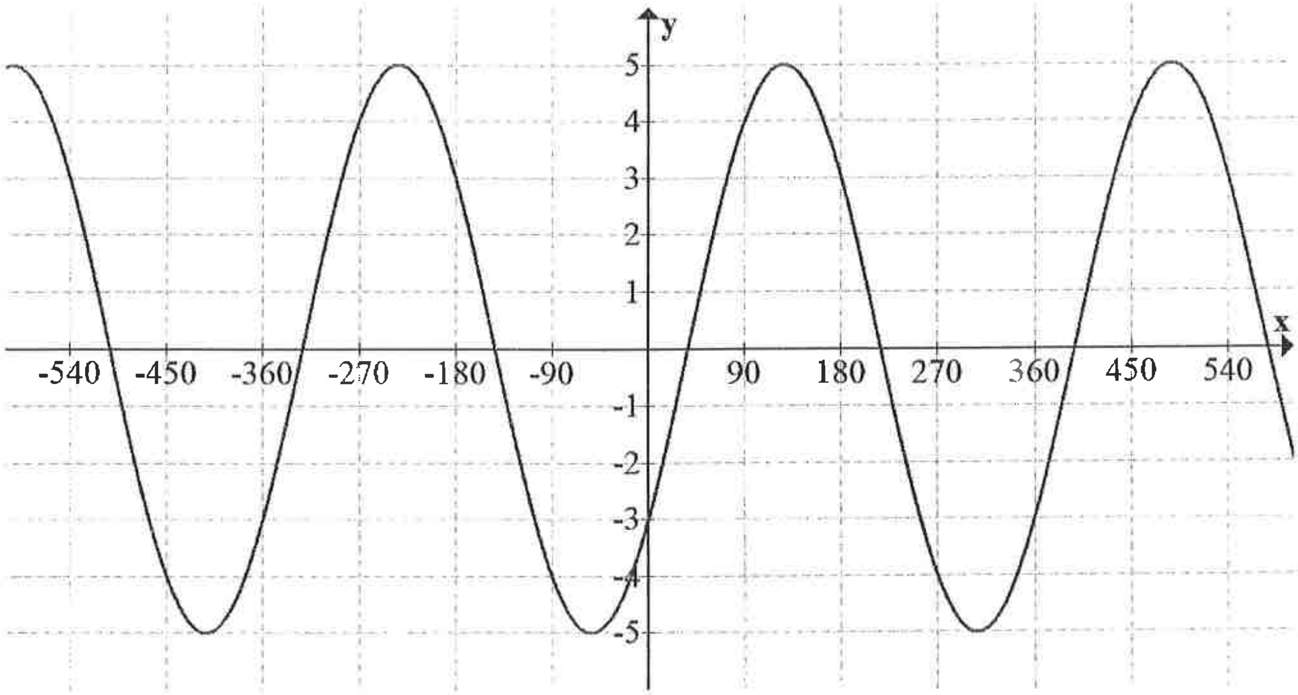
(a) $2\cos(\theta - 30)$ is the cosine curve of magnitude 2, and shifted to the right by 30 degrees:



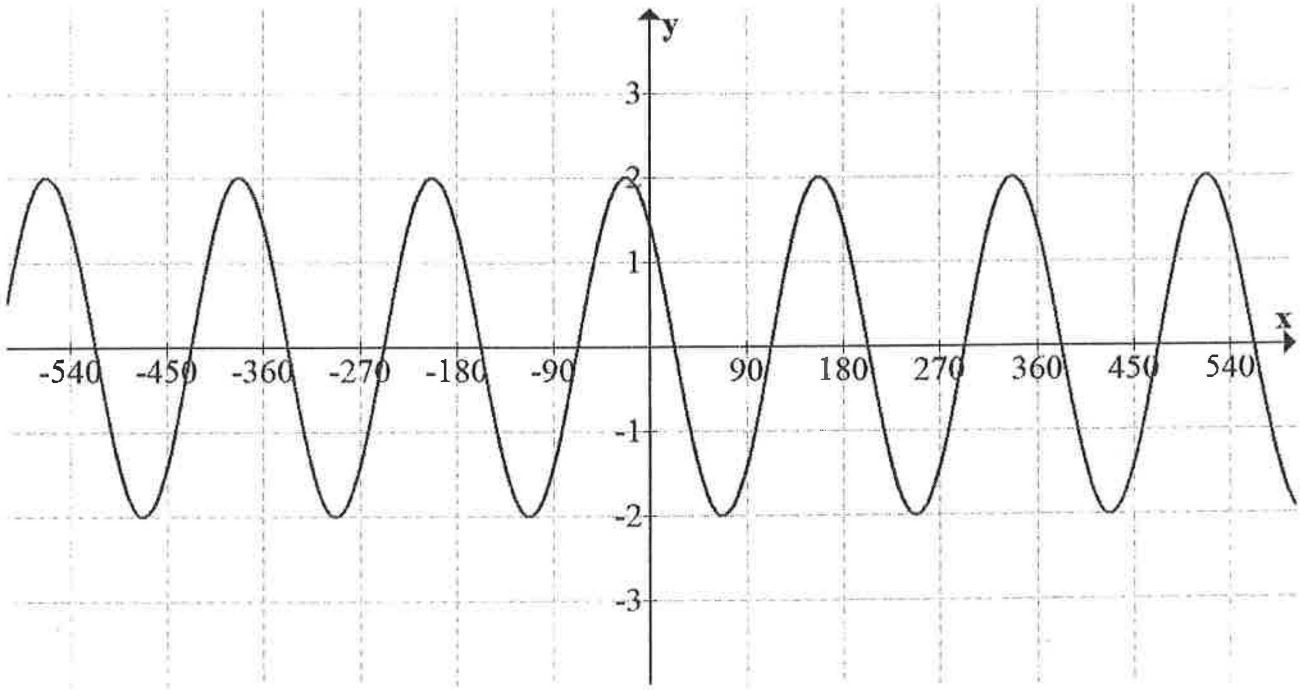
(b) $\sqrt{10}\cos(\theta - 71.57)$ is the cosine curve of magnitude $\sqrt{10}$, and shifted to the right by 71.57 degrees:



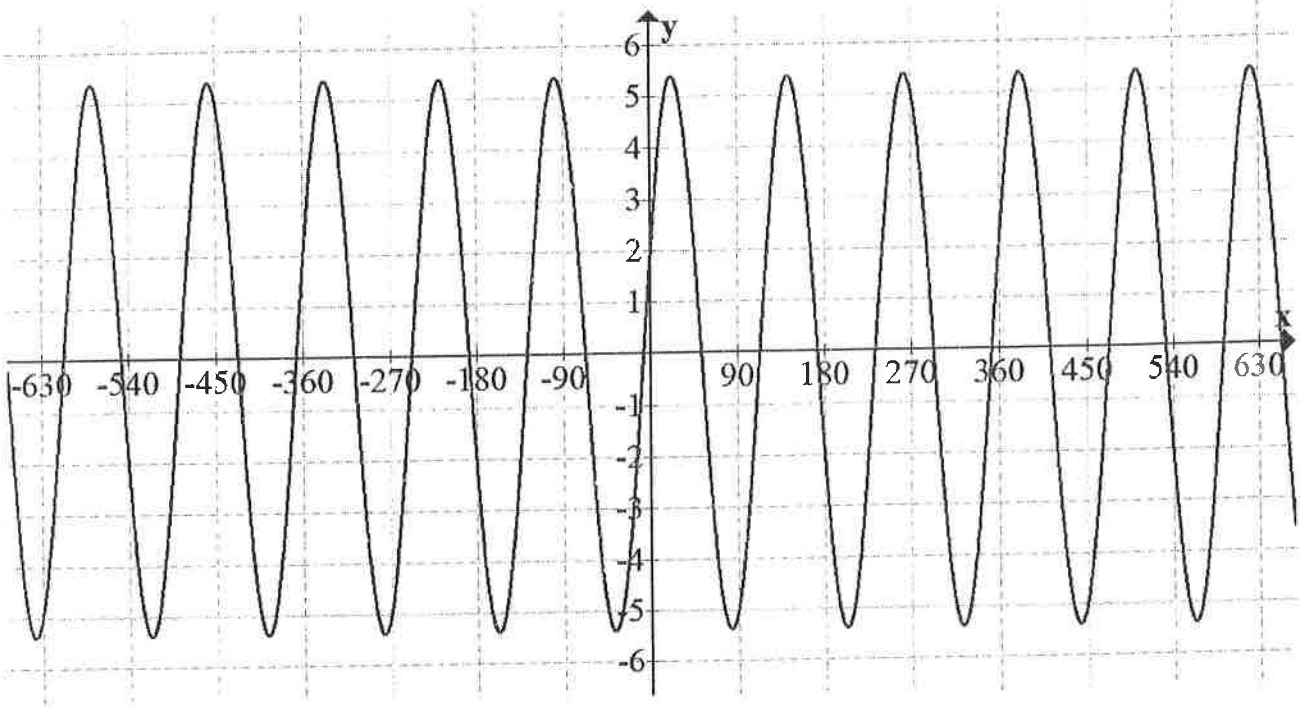
(c) $5\sin(\theta - 36.87)$ is the sine curve of magnitude 5, and shifted to the right by 36.87 degrees:



(d) $2\cos(2\theta + 45) = 2\cos(2[\theta + 22.5])$ is the cosine curve of magnitude 2, and shifted to the left by 22.5 degrees with respect to theta, and has a frequency of 2



(e) $\sqrt{29}\sin(3\theta + 21.8) = \sqrt{29}\sin(3[\theta + 7.26666])$ is the sine curve of magnitude $\sqrt{29}$, and shifted to the left by 7.266666... degrees, and has a frequency of 3.



For (d) and (e) note that shifts/translations of trig functions are measured on θ not on 2θ or 3θ, hence the reason for the factorisation (i.e. to find out the amount θ has shifted, not the amount a multiple of θ has shifted)